# Exploring Rubik's Cube with GAP

In this short lab we will investigate the transformation group of Rubik's cube. Our goal will be twofold. First, we want to gain a better understanding of such concepts as *permutation groups*, *group generators*, and *cyclic subgroups*. Understanding these concepts in the context of a familiar puzzle can make seemingly abstract concepts more accessible. Second, we want to use group theory to find strategic maneuvers of the Cube. For those of you who spent many hours of your childhood playing with this puzzle, perhaps you will also gain from the experience of translating some of the properties you've discovered into the language of group theory.

Before we begin our exploration we will need to introduce some notation. We'll label the sides of each of the 26 "cubelets" according to the following scheme:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 2 | 3 | | | | | | |
| | | | 4 | **Up** | 5 | | | | | | |
| | | | 6 | 7 | 8 | | | | | | |
| 9 | 10 | 11 | 17 | 18 | 19 | 25 | 26 | 27 | 33 | 34 | 35 |
| 12 | **Left** | 13 | 20 | **Front** | 21 | 28 | **Right** | 29 | 36 | **Back** | 37 |
| 14 | 15 | 16 | 22 | 23 | 24 | 30 | 31 | 32 | 38 | 39 | 40 |
| | | | 41 | 42 | 43 | | | | | | |
| | | | 44 | **Down** | 45 | | | | | | |
| | | | 46 | 47 | 48 | | | | | | |

The labeling will allow us to describe motions on the Cube in terms of permutations on the set of labels: {1, 2, 3, . . . , 48}.

**Question 1.** As an example, consider the motion described by:
U := (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19)(12, 36, 28, 20)
(13, 37, 29, 21)(14, 38, 30, 22)(15, 39, 31, 23)(16, 40, 32, 24)(41, 46, 48, 43)(42, 44, 47, 45)
What motion does U represent?

**Question 2.** Suppose F denotes a clockwise turn of the front face of the cube. Describe F as a permutation.

**Question 3.** Determine a "natural" set of generators of the group of transformations of the Cube. That is, find a set of motions that will enable you to create any attainable configuration of the Cube. Then define (in **gap**) each motion as a product of disjoint cycles.

Before continuing it might be useful to define a subgroup $H \leq S_8$ to be the group generated by the set of generators you defined in Question 3.

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gap>  G:=SymmetricGroup(48);
gap>  H:=Subgroup(G,[*,*,*,…]);
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**Question 4.** Next we want to use **gap** to explore what happens as a result of repeatedly applying a particular sequence of motions. Let F denote a clockwise turn of the front face, and R a clockwise turn of the right face. Evaluate the following permutations:
a.) $\{R^2F^2, (R^2F^2)^2, (R^2F^2)^3, \ldots\}$
(Can you find any "strategic maneuvers" among this list? What is the order of $R^2F^2$?)

b.) $\{RFR^{-1}F^{-1}, (RFR^{-1}F^{-1})^2, (RFR^{-1}F^{-1})^3, \ldots\}$
(What is the order of $RFR^{-1}F^{-1}$?)

**Question 5.** When the original Rubik Cube came out, the Ideal Toy Company stated on its package that "there were more than three billion possible states the cube could attain". In his book, *Innumeracy*, J.A. Paulos described this claim as "analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers." Use **gap**'s "Size" command to find the actual size of the Cube's group of transformations.

**Question 6.** a.) Next consider what happens when you repeatedly apply a clockwise turn of the right face followed by a clockwise turn of the front face. That is, evaluate the permutations $\{RF, (RF)^2, (RF)^3, \ldots\}$. What is the order of the cyclic subgroup generated by RF?

b.) Letting B represent a clockwise turn of the back face and L a clockwise rotation of the left face, what is the order of the subgroup generated by RFL?

c.) What is the order of the subgroup generated by RFLB?

d.) Choose your own sequence of motions to explore. Do you think you can find a single sequence that will generate the entire transformation group of the Cube? (That is, do you think the group of motions on the cube is cyclic?)

**Question 7.** Finally we want to examine our initial set of generators. Do you think that the set you defined in Question 3 is the smallest? Explore this idea. See if you can find a smaller set of generators. (Note: In general, it is difficult to find a minimal set of generators of a group. Nonetheless, **gap** gives us the power to experiment with such questions.)